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ON UNCOUNTABLE ABELIAN GROUPS

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ABSTRACT

We continue the investigation from [10], [11], [12] on uncountable abelian groups. This paper tends more to group theory and was motivated by Nunke's statement (in [9]) that Whitehead problem, rephrased properly, is not solved yet.

§0. Introduction

This work continues [10], [12], [13] but here we deal here with more group-theoretic problems, mainly derived from Nunke [9].

In §1 we characterize the Whitehead groups of power $< 2^{\aleph_0}$, assuming Martin Axiom: they are the $\aleph_1$-free groups satisfying possibility II or III from [10]; and, equivalently, they are $\aleph_1$-coseparable or equivalently $\mathrm{Ext}(-, \mathbf{Z}_\omega) = 0$.

In §2 we construct an $\aleph_1$-free group satisfying possibility II which is not strongly $\aleph_1$-free. Hence $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$ implies there is a Whitehead group which is not strongly $\aleph_1$-free.

We also prove (assuming $V = L$ or even $2^{\aleph_0} < 2^{\aleph_1}$) that there is a strongly $\aleph_1$-free, separable, not $\aleph_1$-separable group of cardinality $\aleph_1$. At last we construct an $\aleph_2$-free (hence strongly $\aleph_1$-free) non-separable, non-Whitehead group of cardinality $2^{\aleph_1}$.

In §3 we deal with hereditarily separable groups. If $V = L$ they are just the free groups. (This strengthens the theorem: if $V = 2$, every Whitehead group is free.) But $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$ implies there are non-Whitehead, hereditarily separable groups of cardinality $\aleph_1$. We also prove, assuming $2^{\aleph_0} < 2^{\aleph_1}$, that any hereditarily separable group is strongly $\aleph_1$-free (a little more, in fact).

For notation see Nunke [9] or [13]. $\mathbf{Z}_\omega$ is the direct sum of $\aleph_0$ copies of $\mathbf{Z}$.

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Added in proof. Meanwhile we solve another problem from [9]: ZFC is consistent with the existence of G, $\mathrm{EXT}(G,\mathbf{Z}) = \mathbf{Q}$.

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§1

THEOREM 1.1. $(\mathrm{MA} + 2^{\aleph_0} > \aleph_1)$ *Suppose G is a group of cardinality $\aleph_1$. G is a Whitehead group iff it satisfies possibility* II *or* III *iff G is $\aleph_1$-coseparable iff* $\mathrm{Ext}(G,\mathbf{Z}_\omega) = 0$.

Notice

CONCLUSION 1.2. $(\mathrm{MA} + 2^{\aleph_0} > \aleph_1)$ (1) There are Whitehead groups of cardinality $\aleph_1$ which are not strongly $\aleph_1$-free.

(2) For G a group of cardinality $\leqq \aleph_1$, G is Whitehead iff G is $\aleph_1$-coseparable.

(3) There are non-free $\aleph_1$-coseparable groups of cardinality $\aleph_1$, which are not even $\aleph_1$-separable.

REMARKS. (1) In 1.1, 1.2 we can replace "cardinality $\aleph_1$" by "cardinality $< 2^{\aleph_0}$".

(2) Nunke [9] stated the negation of 1.2(3), but it seemed he was inaccurate.

(3) The proof of 1.1 is similar to [13], §1.

PROOF OF 1.2. (1) Immediate, by 2.1.

(2) Immediate from 1.1.

(3) Immediate by 1.2(1), 1.2(2).

PROOF OF 1.1. Looking at [10], it is clear the only part missing is:

(*) If G is $\aleph_1$-free, $|G| = \aleph_1$, G satisfies possibility I *then* $\mathrm{Ext}(G, Z_\omega) \neq 0$.

Remember (see [5]) that being Whitehead is a hereditary property, G is $\aleph_1$-coseparable iff $\mathrm{Ext}(G,\mathbf{Z}_\omega) = 0$ which implies $\mathrm{Ext}(G,\mathbf{Z}) = 0$, i.e. G is Whitehead and the proof in [10], §3 works for $\mathbf{Z}_\omega$ as well as for $\mathbf{Z}$.

As G satisfies possibility I, there is a countable free G_δ, a_i^l $(l \leqq n(i), i < \omega_1)$ in G, such that:

(i) $\{a_i^l : l \leqq n(i), i < \omega_1\}$ is independent over G_δ;

(ii) $PC_G\langle G_\delta, a_i^0, \cdots, a_i^{n(i)}\rangle / G_\delta$ is not free, w.l.o.g. $n(i) = n(*)$ for every i;

(iii) $PC_G(G_\delta, a_i^0, \cdots, a_i^{n(i)}) / G_\delta$ has no subgroup of smaller rank which is not free.

Let $G_\delta = \bigcup_{m<\omega} G^m$, G^m freely generated by $\{b^0, \cdots, b^{m-1}\}$.

Let G_i^m be $PC(G^m, a_i^0, \cdots, a_i^{n(*)})$. By (ii) above for no i is $G_i^\omega = \bigcup_{m<\omega} G_i^m = PC(G_\delta, a_i^0, \cdots, a_i^{n(*)})$ equal to $\langle G_i^m, G_\delta\rangle$ hence

(iv) For each i for infinitely many $m < \omega$, $G_i^{m+1} \neq \langle G_i^m, G^{m+1}\rangle$. For each $m < \omega$ we define on ω_1 an equivalence relation E_m with countably many equivalence classes:

iE_mj *iff* the mapping f defined by $f \mid G^m = \mathrm{id}$, $f(a_i^l) = a_j^l$ ($l \leqq n(*)$), induces an isomorphism from $PC_G(G^m, a_i^0, \cdots, a_i^{n(*)})$ onto $PC_G(G^m, a_j^0, \cdots, a_j^{n(*)})$. Notice it can induce at most one isomorphism.

As G is $\aleph_1$-free, $PC_G(a_i^0 - a_j^0, \cdots, a_i^{n(*)} - a_j^{n(*)})$ is finitely generated, hence for $i \neq j$ for some m, $\neg\, iE_mj$.

From similar reasons it is clear that E_m has $\leqq \aleph_0$ equivalence classes, and trivially $m < k$ implies that E_k refines E_m. There is $i(*) < \omega_1$ such that for every $i \geqq i(*)$ and m, i/E_m is uncountable (this fails only for countably many i's, so we can choose $i(*)$ big enough).

Claim 1.3. ($\mathrm{MA} + 2^{\aleph_0} > \aleph_1$). *There are an uncountable* $S \subseteq \omega_1 - i(*)$ *and* $k(m) < \omega$ ($m < \omega$) *such that*:

(i) $k(m)$ *is stricly increasing,*

(ii) *for every* $\alpha \in S$, *and* m, $\{j/E_{k(m+1)}: j \in S, jE_{k(n)}\alpha\}$ *has exactly two members,*

(iii) *for every* $i \in S$, *and* m, $\langle G_i^{k(m)}, G^{k(m+1)}\rangle$ *is a proper subgroup of* $G_i^{k(m+1)}$.

Proof. Let us define a partial order P:

$p \in P$ consists of a strictly increasing sequence of natural numbers $\langle k^p(0), \cdots, k^p(n_p)\rangle$, and a finite set V^p of $\omega_1 - i(*)$ such that $k^p(0) = 0$, and for every $i \in V^p$ and $m < n_p$ (letting $k(l) = k^p(l)$)

$$\{j/E_{k(m+1)}: j \in i/E_{k(m)}, j \in V^p\}$$

has exactly two members, and $i \neq j \in V^p$ implies $\neg\, iE_{k(n_p)}j$.

Now $p \leqq q$ if $n_p \leqq n_q$, $\bigwedge_{l \leqq n_p} k^p(l) = k^q(l)$, and $V^p \subseteq V^q$. Clearly, $(\langle 0\rangle, \phi) \in P$.

Fact 1. P satisfies the $\aleph_1$-chain condition.

Let $p(i) \in P$, as we can replace $\{p(i): i < \omega_1\}$ by any uncountable subfamily, w.l.o.g. for every i, $n_{p(i)} = n$, $k^p(l) = k(l)$ ($l \leqq n$), and $V^{p(i)} = \{j^i(l): l < l^*\}$, and $j^i(l)/E_{k(n)}$ depend on l only (not on i). Also w.l.o.g. for some $l^+ \leqq l^*$, $j^i(l) = j(l)$ for $l < l^+$, and $\{j^i(l): l^+ \leqq l < l^*, i < \omega_1\}$ are pairwise distinct (and distinct from $j(l)$ ($l < l^+$)). Now for $l < l^+$ choose $j'(l) \in \omega_1 - i(*) - V^{p(0)} - V^{p(1)}$, $j'(l)E_{k(n)}j(l)$ (possible by an assumption). Choose $k < \omega$ large enough so that $\neg\, j'(l)E_{k(n)}j(l)$ for $l < l^+$, $\neg\, j^0(l)E_kj^1(l)$ for $l^+ \leqq l < l^*$, and $k > k(n)$. Now let $q \in P$ be

$V^q = V^{p(0)} \cup V^{p(1)} \cup \{j'(l): l < l^+\}$, $n_q = n+1$, $k^q(0) = k(0), \cdots, k^q(n) = k(n)$, $k^q(n+1) = k$. It is not hard to check $p^0 \leqq q$, $p^1 \leqq q$, $q \in P$.

FACT 2. $D_i = \{p \in P$: for some $j > i$, $j \in V^p\}$ is dense. We are given $p \in P$, and have to find $q \geqq p$, $q \in D_i$. The proof is like the latter part of Fact 1 (here $V^p = \{j(l): l < l^+\}$).

FACT 3. $D^n = \{p \in P: n_p \geqq n\}$ is dense.

Let $p \in P$; it suffices to show there is $q \geqq p$, $n_q = n_p + 1$ (by iteration). This is proved in Fact 1.

So by Martin Axiom (MA) and $2^{\aleph_0} > \aleph_1$, there is a directed subset A of P, not disjoint to any D_i $(i < \omega_1)$, D^n $(n < \omega)$. So $S = \bigcup_{p \in A} V^p$, $k(n) = k^p(n)$ (for every large enough $p \in A$) exemplify what we want in 1.3.

CLAIM 1.4. *If G^δ, a_i^l, $n(*)$, S are as in* 1.3 *(and before) then G is not a Whitehead group (regardless of whether* $MA + 2^{\aleph_0} > \aleph_1$ *holds).*

NOTE As being Whitehead is a hereditary property we can assume

$$G = PC_G(G_\delta \cup \{a_i^l: l \leqq n(*), i < \omega_1\}).$$

PROOF. We now define by induction on $m < \omega$, a group H^m, and homomorphism h^m such that:

(a) h^m is a homomorphism from H^m onto $\langle \bigcup_{i \in S} G_i^{k(m)} \rangle$ with kernel $\mathbf{Z}$ (note that the range of h^m is not a pure subgroup of G).

(b) h^m, H^m increase with m.

Let $h^m(*a_i^l) = a_i^l$, and $h^m(*a) = a$ for $a \in G_\delta$, $*G^m = (h^m)^{-1}(G^m)$, $*G_\delta = \bigcup_m (h^m)^{-1}(G_\delta)$, $*G_i^m = PC_{H_m}(*G^m, *a_i^0, \cdots, *a_i^{n(*)}) = (h^m)^{-1}(G_i^m)$.

(c) If $i, j \in S$, $iE_{k(m+1)}j$, there is an isomorphism $g_{i,j}^m$: $*G_i^{k(m)} \to *G_j^{k(m)}$, (onto) $g_{i,j}^m \restriction *G^{h(m)} =$ identity, $g_{i,j}^m(*a_i^l) = *a_j^l$ (by the definition of $*G_i^m$ there is at most one such homomorphism).

(d) If $i, j \in S, m > 1$, $iE_{k(m)}j$, but not $iE_{k(m+1)}j$, there is no such $g_{i,j}^{m+1}$.

More specifically, for some $b \in *G_i^{k(m)}$, $c \in *G^{k(m+1)}$, and prime p, $h^{m+1}[(b+c) - (g_{i,j}^m(b) + c))]$ is divisible by p (in G) but $(b+c) - (g_{i,j}^m(b) + c)$ is not divisible by p (in H^{m+1}, hence in every $H^l, l > m$).

Now $h^* = \bigcup h^m$ is a homomorphism from $H = \bigcup H^m$ onto G, so we suppose there is a homomorphism g : Range $h^* \to H$, $h^*g =$ the identity. There is an uncountable $S^* \subseteq S$ such that for all $i \in S^*$, and l

$$*a_i^l - g(a_i^l) = *a_i^l - gh^*(*a_i^l) \in \mathbf{Z}$$

is b^l. Choose $i \neq j$ in S^*, choose m, $iE_{k(m+1)}j$, but not $i\ E_{k(m+1)}j$.

Let b, c as in (d) above. So $b - g^m_{i,j}(b) = (b+c) - (g^m_{i,j}(b) + c)$ is not divisible by p in H. As $b \in {*G}^{k(m)}_i$, for some nonzero integers r, r_l and $a \in {*G}^{k(m)}$, $rb = a + \Sigma_{l \leqq n(*)} r_l * a^l_i$. Clearly $b - g^m_{i,j}(rb)$ is not divisible by kp in H. But $g^m_{i,j} \mid {*G}^{k(m)} = \mathrm{id}$, hence $g^m_{i,j}(a) = a$, hence $rb - g^m_{i,j}(rb) = \Sigma_{l \leqq n(*)} r_l (*a^l_i - *a^l_j)$ is also not divisible by rp. Similarly

$$h^{m+1}(r(b+c) - r(g^m_{i,j}(b)+c)) = h^{m+1}(\Sigma r_l (*a^l_i - *a^l_j))$$

and it is divisible by rp. As $h^{m+1}(*a^l_i) = a^l_i$, $h^{m+1}(*a^l_j) = a^l_j$, $\Sigma r\,(a^l_i - a^l_j)$ is divisible by rp in G so there is $x \in G$, $rpx = \Sigma_{l \leqq n(*)} r_l (a^l_i - a^l_j)$. Hence

$$\begin{aligned} rpg(x) &= g(rpx) \\ &= g\Big(\sum_{l \leqq n(*)} r_l (a^l_i - a^l_j)\Big) \\ &= \sum_{l \leqq n(*)} r_l (g(a^l_i) - g(a^l_j)) \\ &= \sum_{l \leqq n(*)} r_l ((*a^l_i - b^l) - (*a^l_j - b^l)) \\ &= \sum_{l \leqq n(*)} r_l (*a^l_i - *a^l_j). \end{aligned}$$

So $\Sigma_{l \leqq n(*)} r_l (*a^l_i - *a^l_j)$ is divisible by rp (in H). But a little time ago we asserted the opposite. Contradiction.

Conclusion 1.5. (1) Let $\eta_i \in {}^{\omega}2$ $(i < \omega_1)$ be distinct, G_0 is freely generated by $\{x_\eta : \eta \in {}^{\omega >}2$ or $\eta = \eta_i, i < \omega_1\}$, G is generated by G_0 and $(x_{\eta_i} - \Sigma_{l \leqq n} 2^l x_{\eta_i \restriction l})/2^{n+1}$.

Then G is an $\aleph_1$-free non-Whitehead group which is $\aleph_0$-separable. G satisfies possibility I (so is not strongly $\aleph_1$-free).

(2) If above for every $\alpha < \omega$, there are $k_i < \omega$ such that

$$\{\eta_i \mid l : k_i \leqq l < \omega, i < \alpha\}, \{\eta_i \mid l : k_i \leqq l < \omega, i \geqq \alpha\}$$

are disjoint, *then* G is $\aleph_1$-separable (and we can easily find such η_i's).

Proof. Left to the reader.

§2. Examples

Theorem 2.1. *There is an $\aleph_1$-free group of power $\aleph_1$, which is of possibility* II *but not strongly $\aleph_1$-free.*

PROOF. Let S^n $(n < \omega)$ be infinite pairwise disjoint sets of primes, and for each n let S^n_α $(\alpha < \omega_1)$ be infinite pairwise almost disjoint subsets of S^n. Let G^0 be the free group generated freely by $X = \{x^n_\alpha : \alpha < \omega_1, n < \omega\}$, and G^1 its divisible hull (equivalently, the vector space over the rationals generated by X). Let

$$X_\alpha = \{x^n_\beta : \beta < \alpha\}, X^n_\alpha = X_\alpha \cup \{x^m_\alpha : m < n\}.$$

For a subgroup H of G_1, $x = y \ \mathrm{mod}_H n$ means $x - y$ is nz for some $z \in H$.

Let U_n be pairwise disjoint, infinite subsets of ω, such that $m \in U_n$ implies $m > n$. For each $\alpha > 0$, $n < \omega$ we choose an ω-sequence η^n_α such that:

(a) η^n_α is with no repetitions, from X_α and moreover from $\{x^m_\beta : \beta < \alpha, m \in U_n\}$.

(b) If α is a successor η^n_α is included in $X_\alpha - X_{\alpha-1}$.

(c) If α is limit, for each $\beta < \alpha$ only for finitely many $l < \omega$, $\eta^n_\alpha(l) \in X_\beta$.

Now we define our example G. It is the subgroup of G^1 generated by x^n_α $(\alpha < \omega_1, n < \omega)$ and $(x^n_\alpha - \eta^n_\alpha(p))/p$ $(\alpha < \omega_1, n < \omega$ and $p \in S^n_\alpha)$. Let $G_\alpha = PC_G(X_\alpha)$, $G^n_\alpha = PC_G(X^n_\alpha)$.

Clearly G has cardinality $\aleph_1$, so the following facts suffice:

FACT 1. G^n_α is generated by

$$A(\alpha, n) = \{x^m_\beta : x^m_\beta \in X^n_\alpha\} \cup \{(x^m_\beta - \eta^n_\beta(p))/p : x^m_\beta \in X^n_\alpha, p \in S^m_\beta\}.$$

Just prove by induction on (γ, k) that $PC_{\langle A(\gamma,k)\rangle}(X^n_\alpha)$ is generated by the above-mentioned set (i.e., by induction on $\omega\gamma + k$).

FACT 2. If $\{\eta^n_\alpha(p) : p > d\}$ is disjoint from X^m_β then for no $p > d$ and $y \in PC_G(X^m_\beta, x^0_\alpha, \cdots, x^{n-1}_\alpha)$ does $X^n_\alpha = y \ \mathrm{mod}_G\, p$.

If $p \notin S^n_\alpha$ this is easy by Fact 1 (in fact, $y \neq x \ \mathrm{mod}_G p$ for any $y \in G^n_\alpha$). If $p \in S^n_\alpha$, then $\eta^n_\alpha(p) = x^n_\alpha \ \mathrm{mod}_G p$; letting $x^l_\gamma = \eta^n_\alpha(p)$, clearly $n < l$ (see choice of the U_n's) hence $p \notin S^l_\gamma$ (see choice of the S's) hence, by what we said before, $y \neq x^l_\gamma \ \mathrm{mod}_G p$ for $y \in G^m_\beta$ (as clearly $\beta\omega + m < \gamma\omega + l$). So the conclusion is easy for $y \in G^m_\beta$.

Replacing G^m_β by G', change nothing as $S^m_\alpha \cap S^n_\alpha = \emptyset$ for $m < n$.

FACT 3. G is $\aleph_1$-free.

Being a subgroup of G^1, G is torsion free, so it suffices to prove that for any finite $A \subseteq G$, $PC_G(A)$ is finitely generated. However, any generator of G (in the way we define it) is in $PC_G(Y)$ for some $Y \subseteq X$, $|Y| \leq 2$, hence w.l.o.g. A is a finite subset of X. We prove by induction on (α, n) that $PC_{G^n_\alpha}(A \cap X^n_\alpha)$ is finitely generated. In the limit case (i.e., $n = 0$) for some $(\beta, m) < (\alpha, n)$, $A \cap X^n_\alpha \subseteq X^m_\beta$, so as G^m_β is a pure subgroup of G (hence of G^n_α) by its definition $PC_{G^n_\alpha}(A \cap X^m_\alpha) = PC_{G^m_\beta}(A \cap X^m_\beta)$, and our conclusion follows by the induction hypothesis. If $n = m+1$, $x^m_\alpha \notin A$, the same proof applies. So suppose $x^m_\alpha \in A$, choose $p(0) < \omega$, $\beta < \alpha$, $k < \omega$ such that $\{\eta^m_\alpha(l): l \geqq p(0)\}$ is disjoint to X^k_β, but $A \cap X^m_\alpha \subseteq X^k_\beta \cup \{x^0_\alpha, \cdots, x^{m-1}_\alpha\}$. By the induction hypothesis it suffices to prove $PC_G(A \cap X^n_\alpha)/PC_G(A \cap X^m_\alpha)$ is finitely generated, and for this it suffices to prove that, letting $Y = X^k_\beta \cup \{x^0_\alpha, \cdots, x^{m-1}_\alpha\}$, $PC_G(Y \cup \{x^m_\alpha\})/PC_G(Y)$ is finitely generated. This is obvious by Fact 2.

FACT. 4. G is not strongly $\aleph_1$-free.

Suppose $G_1 \subseteq H \subseteq G$, G/H is $\aleph_1$-free, and we shall show $G = H$; as G_1 is countable this clearly suffices. So we prove by induction on (α, n) that $x^n_\alpha \in H$. For $\alpha = 0$, $n < \omega$ this is by assumption. Suppose we have proved it for each $(\beta, m) < (\alpha, n)$, so $X^n_\alpha \subseteq H$. So for each $p \in S^n_\alpha$, $x^n_\alpha = y \bmod_G p$ for some $y \in H$, so $x^n_\alpha/H \in G/H$ is divisible by every $p \in S^n_\alpha$. As G/H is $\aleph_1$-free this implies $x^n_\alpha/H = 0/H$, i.e., $x^n_\alpha \in H$.

FACT 5. G does not satisfy possibility I.

Otherwise there are $\alpha < \omega_1$, and $a^l_i \in G$ $(l \leqq n(i), i < \omega_1)$ such that:

(a) $\{a^l_i: l \leqq n(i), i < \omega_1\}$ is independent over G_α, and

(b) $PC_G(G_i, a^0_i, \cdots, a^{n(i)}_i)/G_\alpha$ is not finitely generated, or equivalently,

(b′) for infinitely many natural numbers d, there are $x = \sum_{l=0}^{n(i)} d^l a^l_i$, $1 = (d^0, \cdots, d^{n(i)})$ (their greatest common divisor), $y \in G_\alpha$ such that $x = y \bmod_G d$. We can assume w.l.o.g.

(c) $\langle a^l_i: l \leqq n(i)\rangle$ has no subgroup of smaller rank which satisfies (b′),

(d) $a^l_i \in G^0$

(Because we can replace $a^0_i, \cdots$ by $da^0_i, \cdots, da^{m(i)}_i$).

For each a^l_i, there is a minimal $Y^l_i \subseteq X$, $a^l_i \in \langle Y^l_i\rangle$. By (a) for some i, $Y^l_i \not\subseteq X_{\alpha+1}$, and choose maximal (β, m) for which $x^m_\beta \in Y = \bigcup_{l \leqq n(i)} Y^l_i$. For some time we fix i. We can replace $\langle a^l_i: l \leqq n(i)\rangle$ by any permutation of it, and by $\langle a^0_i + da^1_i, a^1_i, \cdots, a^{n(i)}_i\rangle$. So in the usual diagonalization of matrices by elementary operations, we can assume $x^m_\beta \in Y^0_i - Y^1_i \cup \cdots \cup Y^{n(i)}_i$, and $a^0_i - d^* x^m_\beta \in \langle Y^0_i - \{x^m_\beta\}\rangle$, $d^* \in \mathbf{Z} - \{0\}$.

By (c) there is a natural number d_0, such that for any d, a, b, $a = \sum d^l a^l_i$, $1 = (d^0, \cdots, d^{n(i)})$, $b \in G_\alpha$, $a = b \bmod_G d$ implies d divides d_0.

By the construction there is a natural number d_1 and a $\gamma < \beta$, $k < \omega$ such that $Y \cap X_\beta \subseteq X^k_\gamma$, $X_\alpha \subseteq X^k_\gamma$, and $\{\eta^m_\beta(l): l \geqq d_1\}$ is disjoint to X^k_γ.

By (b′) there is $d > d^*d_0(d_1!)$, $a = \Sigma_{l \leq n(i)} d^l a_i^l \in \langle a_i^l : l \leq n(i) \rangle$, $b \in G_\alpha$, such that $a = b \bmod_G d$, and $1 = (d^0, \cdots, d^{n(i)})$. As $d > d_0$, clearly $d^0 \neq 0$.

Let d_2 be the greatest common divisor of $d^0 d^*$ and d, and let d_3 be the greatest common divisor of d^0, d and $d_4 = (d^1, \cdots, d^{n(i)})$, so $(d^0, d_4) = 1$ hence $(d_3, d_4) = 1$.

Clearly a/G_β^m is divisible by d, hence $d^0 d^* x_\beta^m / G_\beta^m$ is divisible by d, hence d/d_2 is a product of distinct primes from S_β^m. It is also clear that $\Sigma_{0 < l \leq n(i)} d^l a_i - b$ is divisible in G by d_3 (as $a - b$, $d^0 a_i^0$ are), so as $(d_3, d_4) = 1$ d_3 divides d_0. Now d_2 divides $d_3 d^*$ (by their definitions) which divides $d_0 d^*$.

So some $p \in S_\beta^m$ divides d but not d_2 (hence not $d^0 d^*$) and is $> d_1$.

Let $\eta_\beta^m(p) = x_\zeta^l$, $Y^* = X_\gamma^k \cup \{x_\beta^0, \cdots, x_\beta^{m-1}\}$, then clearly $d^0 d^* x_\beta^m / PC_G(Y^*)$ is divisible by p, hence so are $x_\beta^m / PC_G(Y^*)$, $x_\zeta^l / PC(Y^*)$, but this contradicts Fact 2. (Note that $\omega\gamma + k \leq \omega\zeta + l$.)

THEOREM 2.2. ($\diamondsuit_{\aleph_1}$) *There is a strongly $\aleph_1$-free, $\aleph_0$-separable group of cardinality $\aleph_1$ which is not $\aleph_1$-separable.*

PROOF. We shall define by induction on $\alpha < \omega_1$, a group G_α with universe $\omega(1 + \alpha)$, and for each pure subgroup I of G_α of finite rank, a homomorphism h_I^α such that:

(1) G_α is free, increasing with α, $G_\alpha / G_{\beta+1}$ is free (for $\beta + 1 < \alpha$), as well as G_1/G_0,

(2) h_I^α increases with α, $h_I^\alpha \restriction I$ is the identity, h_I^α is a homomorphism from G_α onto I.

The demands up to now ensure $G = \bigcup_{\alpha < \omega_1} G_\alpha$ will be strongly $\aleph_1$-free, $\aleph_0$-separable of power $\aleph_1$. We shall construct it so that G_0 is not a direct summand. So by the definition of $\diamondsuit_{\aleph_1}$, we can have for each limit $\delta < \omega_1$, a function $h_\delta : G_\delta \to G_0$, such that for any $h: G \to G_0$, $\{\delta : h \restriction G_\delta = h_\delta\}$ is stationary. So it suffices to define $G_{\delta+1}$ in a way that h_δ cannot be extended to a homomorphism from G into G_0, which is the identity on G_0.

So if α is a successor, or $h_\alpha \restriction G_0$ is not the identity or h_α is not a homomorphism into G_0, we can just let $G_{\alpha+1}$ be freely generated by G_α, x_α (there is no problem for $h_I^{\alpha+1}$). In the other case let $\alpha = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$, let p_n be distinct primes, and $\{I_n : n < \omega\}$ be a list of all pure subgroups of G_α of finite rank (in fact $I_n = I_n^\alpha$), and let $\{c_n : n < \omega\}$ be a list of the members of G_0 each appearing $\aleph_0$ times. We shall define by induction on $n < \omega$, β_n, $\alpha_n \leq \beta_n < \alpha$, $\beta_n < \beta_{n+1}$, elements $y_\alpha^n \in G_\alpha - G_{\beta_n}$; we let G_α^n be the group (freely) generated by G_α, x_α, $(x_\alpha - y_\alpha^l)/p_l$ $(l < n)$. We also define in the induction homomorphism $h_{I_l}^{\alpha,n} : G_\alpha^n \to I_l$ $(l < n)$, $h_{I_l}^{\alpha,n}$ increasing with n, and extending $h_{I_l}^\alpha$.

Suppose we have defined $y^m_\alpha, \beta_m (m < n)$ and $h^{\alpha,n}_{I_l}(l < n)$. Choose $\beta_n < \alpha$, $\beta_n > \bigcup_{l<n} \beta_l$, α_n such that $y^0_\alpha, \cdots, y^{n-1}_\alpha \in G_{\beta_n}$, and $I_0, \cdots, I_n \subseteq G_{\beta_n}$.

Clearly G^n_α / G_α is torsion free, of rank 1, and finitely generated, so there is $x^n_\alpha \in G^n_\alpha$, $G^n_\alpha = \langle G^n_\alpha, x^n_\alpha \rangle$, $d_n x^n_\alpha - x_\alpha = b^n_\alpha \in G_\alpha$. For each $m < \omega$ there is at most one homomorphism $h : G^n_\alpha \to G_0$ extending h_α, $h(x_\alpha) = c_m$; call it h^m_α if it exists. Let $k(n)$ be the first $k \geqq n$, such that h^k_α is defined, and there is $z^n_\alpha \in G_\alpha$, $h_\alpha(z^n_\alpha) = h^k_\alpha(x_\alpha) \wedge \bigwedge_{l<n} h^\alpha_{I_l}(z^n_\alpha) = h^{\alpha,n}_{I_l}(z^n_\alpha)$. Choose if possible $t^n_\alpha \in G_\alpha \cap \operatorname{Ker} h_\alpha \cap \bigcap_{l<n} \operatorname{Ker} h^\alpha_{I_l}$ and $\gamma_n(\alpha) < \alpha$, $\gamma_n(\alpha) > \beta_n$, $z^n_\alpha \in G_{\gamma_n(\alpha)}$ such that $t^n_\alpha / G_{\gamma_n(\alpha)}$ is not divisible by p_n. At last choose $s^n_\alpha \in G_0 \cap \bigcap_{l<n} \operatorname{Ker} h_{I_l}$ not divisible by p_n (this is a pure subgroup of G_0, and $G_0/(G_0 \cap \bigcap_{l<n} \operatorname{Ker} h_{I_l})$ has finite rank, so such s^n_α exists).

If $k(n)$, z^n_α and t^n_α are defined, we let $y^n_\alpha = z^n_\alpha + t^n_\alpha + s^n_\alpha$, and continue; otherwise we stop. If we continue it is easy to check $h^{\alpha,n}_{I_l}$ $(l < n)$ has one (and only one) extension $h^{\alpha,n+1}_{I_l}: G^{n+1}_\alpha \to I_l$, and h_α has no extension $h : G^{n+1}_\alpha \to G_0$, $h(x_\alpha) = c_{k(m)}$, and we can define $h^{\alpha,n+1}_{I_n}$.

If our induction stops at some n, we behave as for a successor α, and if we finish it, $G_{\alpha+1}$ is generated by G_α, x_α, $(x_\alpha - y^n_\alpha)/p_n$, and then we let $h^{\alpha+1}_{I_l} = \bigcup_{n \geqq 1} h^{\alpha,n}_{I_l}$. In the other cases ($I \subseteq G_{\alpha+1}, I \not\subseteq G_\alpha$, or the induction stops) there is no problem to define $h^{\alpha+1}_I$.

If our induction is finished it is not hard to check that h_α has no extension $h : G_{\alpha+1} \to G_0$.

The only point we have to show is that if $h : G \to G_0$ is a homomorphism, and $h \mid G_0 =$ the identity, then for some δ, $h_\delta = h \mid G_\delta$, and the induction is finished.

However, $C_1 = \{\delta < \omega_1$: for every pure $I_0, \cdots, I_n \subseteq G_\delta$, of finite rank there is $\gamma < \delta$ such that $h(x_\delta) = h(x_\gamma)$, $h_{I_0}(x_\delta) = h_{I_0}(x_\gamma), \cdots, h_{I_n}(x_\delta) = h_{I_n}(x_\gamma)\}$ is closed and unbounded.

Similarly, $C_2 = \{\delta < \omega_1$: for every $\beta < \delta$, pure $I_0, \cdots, I_n \subseteq G_\delta$ of finite rank there are successors $\gamma(1) < \gamma(2) < \delta$, $\beta < \gamma(1)$, $h(x_{\gamma(1)} - x_{\gamma(2)}) = h_{I_0}(x_{\gamma(1)} - x_{\gamma(1)}) = \cdots = h_{I_n}(x_{\gamma(1)} - x_{\gamma(2)}) = 0\}$ and $S = \{\delta < \omega_1 : h \mid G_\delta = h_\delta\}$ is stationary.

So there is $\delta \in S \cap C_1 \cap C_2$, and for it the induction is finished, i.e., for every n, $z^n_\delta, t^n_\delta, \gamma_n(\delta), s^n_\delta$ exist.

We can improve this to:

THEOREM 2.3. *The last theorem holds even assuming only* $2^{\aleph_0} < 2^{\aleph_1}$.

PROOF. This time we use the fact that ω_1 is not small (see Devlin and Shelah [3]). We this time define by induction on $\alpha < \omega_1$ for $\eta \in {}^\alpha 2$, a free group G_η with universe $\omega(1 + \alpha)$, and for each pure subgroup I of G_η of finite rank, a projection $h_l^\eta: G_\eta \to I$ onto I, both increasing by $\lhd$, such that $G_\eta / G_{\eta|(\beta+1)}$, $G_\eta / G_{\langle\,\rangle}$ are free (where $\beta < l(\eta)$), $G_{\langle\,\rangle}$ has rank $\aleph_0$ and:

(*) for limit $\delta < \omega_1$, $\eta \in {}^{\delta}2$ there are no projections $h_l: G_{\eta\hat{}\langle l\rangle} \to G_0$ onto G_0 $(l = 0, 1)$, $h_0 \restriction G_\eta = h_1 \restriction G_\eta$, $h_l \restriction G_0 = \mathrm{id}$.

Now by Θ (see [3], §6) for some $\eta \in {}^{(\omega_1)}2$, G_η does not have a projection onto G_0, then this is the group we are trying to construct.

For the construction, let $\beta_n < \beta_{n+1} < \delta$, $\bigcup \beta_n = \delta$, $y_n \in G_{\eta|\beta_{n+1}}$, $y_n / G_{\eta|\beta_n}$ not divisible by p_n, $G_{\eta\hat{}\langle l\rangle} = \langle G_\eta, x_{\eta\hat{}\langle l\rangle}, (x_{\eta\hat{}\langle l\rangle} - y_n - x_0 d_n^l)/p_n\rangle$, where $\langle x_0\rangle = p \subset_{G_{\langle\,\rangle}} (x_0) \subseteq G_{\langle\,\rangle}$. We have to choose the d_n^l, so that $(y_n + x_0 d_n^l)/G_{\eta|\beta_n}$ is not divisible by p_n, and to destroy all possible pairs $\langle h_0(x_{\eta\hat{}\langle 0\rangle}), h_1(x_{\eta\hat{}\langle 1\rangle})\rangle$ (from $G_{\langle\,\rangle}$.)

THEOREM 2.4. *There is a strongly $\aleph_1$-free group which is not $\aleph_0$-separable of power $2^{\aleph_1}$. Moreover, there is a $\aleph_2$-free, strongly $\aleph_1$-free not Whitehead group of cardinality $2^{\aleph_1}$.*

REMARK. This theorem answers negatively a question of Eklof [4] as to whether the class of $\aleph_1$-separable $\aleph_0$ groups is definable in L_{∞,ω_1} (see [4] p. 106, paragraph before theorem 2.11).

PROOF. Let $\lambda = 2^{\aleph_0}$.

Let H_0, H_1 be free groups of cardinality $\aleph_1$, such that $H_0 \subseteq H_1$, H_1/H_0 is $\aleph_1$-free but not a Whitehead group, exists by 1.5. Let $\{z_i^l: i < \omega_1\}$ freely generate H_l $(l = 0, 1)$.

Let G_0 be freely generated by x_η $(\eta \in {}^{(\omega_1>)}\lambda)$ and G be generated by $G_0 \cup \{y_\eta^i: i < \omega_1, \eta \in {}^{\omega_1}\lambda\}$ freely except that:

(*) there are embeddings $h_\eta: H_1 \to G$, $h_\eta(z_i^0) = x_{\eta|i}$, $h_\eta(z_i^1) = y_\eta^i$, for $\eta \in {}^{(\omega_1)}\lambda$.

Let for $\eta \in {}^{(\omega_1 \geq)}\lambda$, $G_\eta = \langle x_{\eta|\alpha}: \alpha < l(\eta)\rangle$, $H_\eta = \langle y_\eta^\alpha: \alpha < \omega_1\rangle$.

FACT 1. G is $\aleph_2$-free.

Any subgroup G^* of G of power $\leqq \aleph_1$ is contained in $\langle H_\eta: \eta \in S\rangle$ for some $s \subseteq {}^{(\omega_1)}\lambda$, $|S| \leqq \aleph_1$, and let $S = \{\eta_i: i < \omega_1\}$. We can define by induction on i, $\alpha_i < \omega_1$, such that $B_i = \{\eta_i \restriction \beta: \alpha_i \leqq \beta < \omega_1\}$ are pairwise disjoint. Let us define

$$I_0 = \left\langle \left\{ x_\nu: \nu = \eta_i \restriction \alpha \text{ for some } i,\ \alpha < \omega_1,\ \nu \notin \bigcup_{i<\omega_1} B_i \right\} \right\rangle,$$

$$I_i = \left\langle I_0, \bigcup_{\beta<i} H_\beta \right\rangle.$$

Clearly I_i $(i < \omega_1)$ is an increasing continuous sequence of subgroups of G whose union is $\langle H_{\eta_i}: i < \omega_1\rangle$. So it suffices to prove I_0, I_{i+1}/I_i are free.

I_0 is free as a subgroup of G_0.

I_{i+1}/I_i is isomorphic to $H_{\eta_i}/G_{\eta_i|\alpha_i}$ which is easily verified to be free.

We now find a group G_0^+, $\mathbf{Z} \subseteq G_0^+$, and a homomorphism g_0 from G_0^+ onto G_0, $\operatorname{Ker} g_0 = \mathbf{Z}$. Then by induction on $\alpha < \omega_1$, for each $\eta \in {}^{(\alpha+1)}\lambda$ we assign $f_\eta: G_{\eta|\alpha} \to G^+_{\eta|\alpha}$ (where $G^+_\nu = g_0^{-1}(G_\nu)$) such that $\nu < \eta \Rightarrow f_\nu \subseteq f_\eta$, $f_\eta g_{\eta|\alpha} = \mathbf{1}_{G_{\eta|\alpha}}$ (where $g_{\nu|\alpha} = g_0 \mid G^+_{\nu|\alpha}$) and for every $f: G_\eta \to G^+_\eta$, $fg_\eta = \mathbf{1}_{G_\eta}$ extending f_η, for some $\alpha < \lambda$ $(= 2^{\aleph_0})$, $f = f_{\eta^\wedge\langle\alpha\rangle}$.

Now for $\eta \in {}^{(\omega_1)}\lambda$ we define H^+_η and a homomorphism g^η from H^+_η onto H_η, $G^+_\eta \subseteq H^+_\eta$, $g_\eta \subseteq g^\eta$, $\operatorname{Ker} g^\eta = \mathbf{Z}$ such that $f_\eta = \bigcup_{\alpha<\omega_1} f_{\eta|(\alpha+1)}$ cannot be extended to a homomorphism from H_η into H^+_η, $f_\eta g_\eta = \mathbf{1}_{G_\eta}$ (this is possible as H_η/G_η is not a Whitehead group).

Now we define G^+, g such that $H^+_\eta \subseteq H$ for $\eta \in {}^{(\omega_1)}2$, g extend every g^η ($\eta \in {}^{(\omega_1)}2$) and g is a homomorphism from G^+ onto G, $\operatorname{Ker} g = \mathbf{Z}$ (no problem as there was no "connection" between the H_η's except through G_0). Now G^+, g exemplify G is not a Whitehead group. For suppose $f: G \to G^+$, $fg = \mathbf{1}_G$, then define by induction on $\alpha < \omega_1$, $\gamma(\alpha) < \lambda$ such that

$$f \mid G_{\langle\gamma(i): i<\alpha\rangle} = f_{\langle\gamma(i): i\leqq\alpha\rangle},$$

let $\eta = \langle\gamma(\alpha): \alpha < \omega_1\rangle$, so $f \supseteq \bigcup_{\alpha<\omega_1} f_{\eta|(\alpha+1)}$, so $f \mid H_\eta$ contradict the choice of g^η.

So G is $\aleph_2$-free and not Whitehead, G^+ is $\aleph_2$-free and not separable ($\mathbf{Z}$ is not a direct summand). We finish noting that by [11], $\aleph_2$-free implies strongly $\aleph_1$-free.

THEOREM 2.5. (1) *In the example from Theorem* 2.4 *the G we construct is $\aleph_1$-separable, provided that each H_η/G_η is $\aleph_1$-separable.*

(2) *We can make G not hereditarily separable.*

PROOF. (1) Left to the reader.

(2) We choose $G'_{\langle\ \rangle} \subseteq G_{\langle\ \rangle}$, $G/G'_{\langle\ \rangle}$ isomorphic to $\mathbf{Z}_p^{(\infty)}$ (p a prime), $x_0 \in G_{\langle\ \rangle} - G'_{\langle\ \rangle}$, $px_0 \in G'_{\langle\ \rangle}$, and then $G'_\eta \subseteq G_\eta$ ($\eta \in {}^{\omega_1 \geqq}\lambda$) increasing with η (by $\lhd$), $x_0 \notin G'_\eta$, G_η/G'_η isomorphic to $\mathbf{Z}_p^{(\infty)}$, and for each $\eta \in {}^{\omega_1}\lambda$ we have a projection h_η of G'_η onto $p\mathbf{Z}$ where we identify $\mathbf{Z}$ with $\langle x_0\rangle$.

We now have to define $H'_\eta \subseteq H_\eta$, $H'_\eta \cap G_\eta = G'_\eta$, $H_\eta/H'_\eta \cong \mathbf{Z}_p^{(\infty)}$, so that h_η cannot be extended to a projection of H'_η onto $p\mathbf{Z}$. This is done as in 1.5–1.4.

§3

DEFINITION 3.1. (1) An abelian group G is hereditarily separable if it is $\aleph_0$-free and for every subgroup G', and finitely generated pure subgroup H of

G', H is a direct summand of G'. We can replace "finitely generated" by "isomorphic to $\mathbf{Z}$" (see [5] or [9]).

REMARK. (2) The hypothesis "for every regular λ and stationary $S \subseteq \lambda$ the weak diamond holds" (see [3]) is sufficient for Theorem 3.1 (see the proof of 3.5 and then change the proof of 3.1 accordingly).

THEOREM 3.1. *Suppose $V = L$, or even that for every regular λ and stationary $S \subseteq \lambda$ $\diamondsuit_S$ holds.*

Then every hereditarily separable torsion free group is free.

Before proving this theorem we first establish two facts.

FACT 1. The following are equivalent where $H_1 \subseteq H_2$ and I are abelian groups:

(a) every $h: H_1 \to I$ has at most one extension to $h': H_2 \to I$,

(b) if $h: H_2 \to I$, $h \mid H_1 = \mathbf{0}_{H_1}$ then $h \mid H_2 = 0_{H_2}$,

(c) if $h: H_2/H_1 \to I$, then $h = 0$.

PROOF OF FACT 1. If (a) fails, h_1, h_2: $H_2 \to I$ extend h and $h_1 \neq h_2$, then $h_1 - h_2$ shows that (b) fails. If (b) fails, h exemplifies this, the mapping $x/H_1 \to h(x)$ (well defined as $H_1 \subseteq \operatorname{Ker} h$) shows (c) fails. If (c) fails and h exemplifies it, let $h_1(x) = 0$ $(x \in H_2)$, $h_2(x) = h(x/H_1)$, so $h_1 \neq h_2$: $H_2 \to I$ extend $\mathbf{0}_H$, thus showing that (a) fails.

FACT 2. If $I = \mathbf{Z}$, or even $\aleph_0$-free, H is not free, of finite rank and every subgroup of smaller rank is free, and is torsion free, then every $h: H \to I$ is zero.

PROOF OF FACT 2. Let $h \neq 0$. The range of h is a subgroup of I of finite rank, so w.l.o.g. I has finite rank, hence is free; let h_0: $I \to \mathbf{Z}$ be such that $h_0 h \neq 0$ (easy). So $H_1 = \operatorname{Ker}(h_0 h)$ is a subgroup of H of rank $<$ rank H, hence H_1 is free, and h^*: $H/H_1 \to \mathbf{Z}$ defined by $h^*(x/H_1) = h_0 h(x)$ is a well defined homomorphism, and $\neq \mathbf{0}$, $\operatorname{Ker} h^* = 0$. So h^* is an embedding, but H/H_1 is not finitely generated, as H is not free, contradiction.

PROOF OF THEOREM 3.1. Let G_0 be any torsion-free, hereditarily separable group and H_0 be a pure free subgroup of rank $\aleph_0$.

Let p be any prime, $\{x_n: n < \omega\}$ generate freely H_0, and let H_0' be the subgroup of H_0 generated by $\{p^{n+1}x_n: n < \omega\} \cup \{x_n - px_{n+1}: n < \omega\}$. (So H_0/H' is isomorphic to $\mathbf{Z}_p^{(\infty)}$.) We prove by induction on λ:

$(*)_\lambda$ Suppose G is torsion free, H a pure subgroup of G, G/H has rank $\leqq \lambda$, $H' \subseteq H$, $H/H' \cong \mathbf{Z}_p^{(\infty)}$, and more specifically $H = \langle H', \cdots, x_n, \cdots \rangle_{n<\omega}$, $px_0 \in H'$,

$x_n - px_{n+1} \in H'$, $x_0 \notin H'$, and G/H is not free. We identify $\mathbf{Z}$ with $\langle x_0 \rangle \subseteq H$, so $p\mathbf{Z}$ is a pure subgroup of H'.

Then:

(a) If h is a projection of H' onto $p\mathbf{Z}$, we can find $G' \subseteq G$, $G = \langle G', \cdots, x_n, \cdots \rangle_{n<\omega}$, $H' = G' \cap H$, such that h cannot be extended to a projection of G' onto $p\mathbf{Z}$.

(b) If in addition G is $|H|^+$-free we can in (a) find G' suitable for all h.

Clearly (b) gives our conclusion (with G_0, H_0, H' for G, H, H') for uncountable G. We can in fact weaken the hypothesis of (b) to: There is no $G^* \subseteq G$, $|G^*| \leqq |H|$, G/G^* free.

We prove it by induction on λ.

Choose G_1, $H \subseteq G_1 \subseteq G$, such that G_1/H is not free, and the rank of G_1/H is minimal. It suffices to prove $(*)_\lambda$ for G_1, because if G_1' is as required (for G_1), let G' be a maximal subgroup of G such that $G' \cap G_1 = G_1'$ (equivalently, $G_1' \subseteq G'$, $x_0 \notin G'$). Notice the rank of G_1/H is $\leqq \lambda$, and G_1 is $|H|^+$-free if G is $|H|^+$-free.

By [11], the rank κ of G_1/H is finite, or a regular uncountable cardinal.

Case 1. κ finite.

Let $z_1/H, \cdots, z_\kappa/H$ be a maximal independent set in G_1/H, and w.l.o.g. $\langle z_1/H, \cdots, z_{\kappa-1}/H \rangle$ generate a pure subgroup of G_1/H. Let I be a maximal subgroup of G_1, such that $I \cap H = H'$, $z_1, \cdots, z_{\kappa-1} \in I$, $pz_\kappa \in I$ but $z_\kappa + lx_0 \notin I$ for every l, $0 \leqq l < p-1$.

Subcase 1*A*. I/H' is not free.

Clearly I/H' has rank κ, and every subgroup of smaller rank is free, hence h has a unique extension h^* to a projection of I onto $p\mathbf{Z}$.

Choose a number $l \in \{0, 1\} \subseteq \mathbf{Z}$ such that $h^*(pz_\kappa) + pl$ (in $\mathbf{Z}$) is not divisible by p^2 (in $\mathbf{Z}$), and let $G'' = \langle I, z_\kappa + lx_0 \rangle$, G' be a maximal subgroup of G_1, $G'' \subseteq G'$, $G' \cap H = H'$. G' is as required, because if h' is a projection from G' onto $p\mathbf{Z}$ as required, necessarily $h' \supseteq h^*$. So (remembering $1 = x_0$)

$$ph'(z_\kappa + l) = h'(pz_\kappa + pl) = h'(pz_\kappa) + h'(pl) = h^*(pz_\kappa) + h'(pl)$$

$$= h^*(pz_\kappa) + pl.$$

All numbers are in $\mathbf{Z}$, but moreover $h'(z_\kappa + l) \in p\mathbf{Z}$, so $h^*(pz_\kappa) + pl$ is divisible by p^2 (in $\mathbf{Z}$), contradiction.

The other conditions on G' are easy to check.

Subcase 1*B*. I/H' is free.

It is clear that if q is a prime $\neq p$, $z \in G_1$, $qz \in I$, then $z \in I$ (by the maximality of I). Also G_1/I is torsion (as $\langle H', z_1, \cdots, z_{\kappa-1}, pz_\kappa\rangle \subseteq I$), so it is a p-group. Hence also $G_1/(I+H)$ is a p-group. As $I \cap H = H'$ clearly $I/H' \cong (I+H)/H$, so as they are free, $(I+H)/(H+\langle z_1, \cdots, z_{\kappa-1}, pz_\kappa\rangle)$ is finite. So if $G_1/(I+H)$ is finite then $G_1/(H+\langle z_1, \cdots, z_{\kappa-1}, pz\rangle)$ is finite. Hence $G_1/(I+H)$ is finitely generated, hence free, contradiction. So $G_1/(H+I)$ is not finite. Now $G_1/(H+I)$ has rank 1 (it cannot have rank 0, as it is not finite; if it has rank >1, then there is $y \in G_1-(H+I)$, $z_\kappa/(H+I)$ not in the subgroup that $y/(H+I)$ generate). As $G_1/(H+I)$ is a p-group, we can assume $py \in H+I = \langle I, x_0 \cdots\rangle$. So $py = lx_m + y'$ for some $m>0$, $l \in \mathbf{Z}$, $y' \in H$. Now we can replace y by $y - lx_m$, so now $py \in I$. Let $I' = \langle I, y\rangle$. Now $z_\kappa + lx_0 \notin I'$ (as otherwise $z_k/(H+I) \in (I'+H)/(I+H)$, contradicting the choice of y). Also $x_0 \notin I$ (as otherwise $x_0 - ly \in I$, so (as $x_0 \notin I$) $(l,p)=1$, and then $ly \in (H+I)$, which together with $(p,l)=1$ implies $y \in (H+I)$, contradiction). Hence $I' \cap H = H'$. So I is not maximal, contradiction. Hence the rank of $G_1/(H+I)$ is 1.

The only (up to isomorphism) infinite p-group of rank 1 is $\mathbf{Z}^p_\infty$, which is p-divisible. We show that G_1/I (which is a p-group) is p-divisible. Let $y/I \in G_1/I$. As $G_1/(H+I)$ is p-divisible, there is $y_1 \in G_1$, $y - py_1 \in H+I$, so as $H' \subseteq I$, $y - py_1 = lx_k + y_2$ for some $y_2 \in I$, l and k. Now $y - p(y_1 + lx_{k+1}) = (y-py_1) - lpx_{k+1} = lx_k + y_2 - lx_k + l(x_k - px_{k+1}) = y_2 + l(x_k - px_{k+1}) \in I + H' = I$. So y/I is divisible by p. Now h has only $\aleph_0$ extensions to a homomorphism from G_1 into $\mathbf{Q}$ (the only freedom we have is the images of $z_1, \cdots, pz_\kappa$; remember $\mathbf{Z} \subseteq \mathbf{Q}$, and we identify $1 \in \mathbf{Z}$ and x_0).

Let us enumerate them h^k $(k<\omega)$. Now we define $t_k \in G_1$ such that $t_0 \in G_1$, $t_0 \notin I$, $pt_0 \in I$, $pt_{k+1} - t_k \in I$, $x_0 \notin \langle I, t_0, \cdots, t_k\rangle$.

Let $t_0 = z_\kappa$ (check $x_0 \notin \langle I, z\rangle$ by I's definition).

If t_k is defined, choose $t^0_{k+1} \in G_1$, $pt^0_{k+1} - t_k \in I$ (by the p-divisibility of G_1/I). Choose $l \in \{0,1\}$ such that $h^k(t^0_{k+1} + lx_0)$ is not in $p\mathbf{Z} = \langle px_0\rangle$ (possible as $x_0 \notin p\mathbf{Z}$), and let $t_{k+1} = t^0_{k+1} + lx_0$.

Now let $G' = \langle I, t_0, \cdots, t_k, \cdots\rangle$.

Case 2. κ regular uncountable cardinal.

So let G_1 be $PC_{G_1}(H \cup \{a_i : i<\kappa\})$, $\{a_i : i<\kappa\}$ independent over G. Let $\alpha(i) < \kappa$ $(i<\kappa)$ be increasing and continuous. Let G^j be $PC_{G_1}(H \cup \{a_i : i < \alpha(j)\})$. Clearly $S = \{\alpha < \kappa :$ for some $\beta > \alpha$, G^β/G^α is not free$\}$ is stationary, so w.l.o.g. $\alpha \in S$ implies $G^{\alpha+1}/G^\alpha$ is not free. Trivially the rank of $G^{\alpha+1}/G^\alpha$ is $<\kappa$. Clearly any homomorphism from G^α into $\mathbf{Q}$ extending h is determined by the images of the a_i's (and vice versa — every function from $\{a_i : i<\kappa\}$ to $\mathbf{Q}$ can be extended to such homomorphism). As by a hypothesis, $\diamondsuit_S$ holds, there are homomorphisms $h_\alpha : G^\alpha \to \mathbf{Q}$ $(\alpha \in S)$ such that:

(i) for any homomorphism $h': G_1 \twoheadrightarrow \mathbf{Q}$, $h \subseteq h'$, $\{\alpha \in S: h' \upharpoonright G^\alpha = h_\alpha\}$ is stationary.

(ii) If $|H| < \kappa$ (which occurs in (b)) we can omit the demand $h \subseteq h'$.

Now we can define by induction on $\alpha < \lambda$, groups $H^\alpha \subseteq G^\alpha$, H^α increasing with α, $x_0 \notin H^\alpha$, $G^\alpha = \langle H^\alpha, x_0, x_1, \cdots \rangle$, and if $\alpha \in S$, h_α a projection from H^α onto $p\mathbf{Z}$, *then* h_α cannot be extended to a projection from $H^{\alpha+1}$ onto $p\mathbf{Z}$.

For $\alpha = 0$, $H^\alpha = H'$; for α limit $H^\alpha = \bigcup_{\beta<\alpha} H^\beta$; for α successor, if h_α is a projection from G^α onto $p\mathbf{Z}$ use the induction hypothesis, otherwise it is trivial. Now we define G' as $\bigcup_\alpha H^\alpha$.

So we finish Case 2, hence the theorem.

Definition 3.2. For a natural number m (>1) a group G is called m-hereditarily separable *if* G is $\aleph_1$-free and for any homomorphism $h: G \to \mathbf{Q}_m/\mathbf{Z}$ (where $\mathbf{Q}_m$ is the additive subgroup of $\mathbf{Q}$ generated by $1/m, 1/m^2, \cdots, 1/m^k, \cdots$) and pure subgroup I^* of G isomorphic to $\mathbf{Z}$, there is a homomorphism $g: \operatorname{Ker} h \to I^* \cap \operatorname{Ker} h$, $g \upharpoonright (I^* \cap \operatorname{Ker} h) =$ the identity.

Claim 3.2. *The following conditions on a group G are equivalent*:

(a) *G is hereditarily separable.*

(b) *G is m-hereditarily separable for every natural number m* (>1).

(c) *G is p-hereditarily separable for every prime p.*

Proof. See later.

Theorem 3.3. $(\mathrm{MA} + 2^{\aleph_0} > \aleph_1)$ *Let G be an* $\aleph_1$*-free group of cardinality* $< 2^{\aleph_0}$. *Then the following conditions are equivalent* (*we can erase the "for every p"*):

(i) *G is hereditarily separable, i.e., p-hereditarily separable for every prime p.*

(ii) *For every p, and finite subsets* $A_i \subseteq G$ $(i < \omega_1)$, *there are* $S_0 \subseteq \omega_1$, $n < \omega$, $a^i_l \in G$ $(i \in S_0, l = 1, \cdots, n\}$, S_0 *uncountable*, $A_i \subseteq \langle a^i_1, \cdots, a^i_n \rangle$ (*for* $i \in S_0$) *such that for every uncountable* $S_1 \subseteq S_0$ *there are* $i \neq j \in S_0$ *such that*:

$$(\alpha) \quad \begin{aligned} &PC(a^i_1, \cdots, a^i_n, a^j_1, \cdots, a^j_n) = \\ &\langle PC(a^i_1, \cdots, a^i_n), PC(a^j_1, \cdots, a^j_n), PC(a^i_1 - a^j_1, \cdots, a^i_n - a^j_n) \rangle, \end{aligned}$$

(β) $\sum_{l=1}^n k^i_l a^i_l = \sum_{l=1}^n m^j_l a^j_l$ *implies* $k^i_l = m^j_l$ *for* $l = j, \cdots, n$,

(γ) *no element of* $PC(a^i_1 - a^j_1, \cdots, a^i_n - a^j_n)/\langle a^i_1 - a^j_1, \cdots, a^i_n - a^j_n \rangle$ *has order p.*

(iii) *For no countable pure subgroup* $G_0 \subseteq G$ *are there* a^i_l $(l \leqq n_i, i < \omega_1)$ *such that*:

(α) *in* G/G_0, *the set* $\{a^i_l/G_0: l \leqq n_i, i < \omega_1\}$ *is independent,*

(β) *in* $PC(G_0 \cup \{a^i_l : l \leqq n_i\})/PC(G_0 \cup \{a^i_l : l < n_i\})$ *there are elements* $t_m \neq 0$, $pt_{m+1} = t_m$ (for $m < \omega$).

PROOF OF CLAIM 3.2. (a) $\Rightarrow$ (b). Let $h: G \to \mathbf{Q}_m/\mathbf{Z}$, $I^* \subseteq G$ be as in (b), and let $H = \operatorname{Ker} h$. Clearly $I^* \cap H$ is a pure subgroup of H isomorphic to $\mathbf{Z}$, so by (a) there is $g: H \to I^* \cap H$, $g \restriction (I^* \cap H) =$ the identity.

(b) $\Rightarrow$ (a). Let H be a subgroup of G (not necessarily pure), I^* a pure subgroup of H of rank 1 (equivalently, isomorphic to $\mathbf{Z}$). It suffices to find $g: H \to I^*$, $g \restriction I^* =$ the identity. Clearly we can replace H by any H', $H \subseteq H' \subseteq G$, $H \cap PC_G(I^*) = I^*$, so w.l.o.g. H is maximal with respect to those properties. Clearly $PC_G(I^*)$ is of rank 1, hence isomorphic to $\mathbf{Z}$, and let x_0 generate it; $m = \min\{n: nx_0 \in I^*\}$. By the maximality of H, G/H has no subgroup disjoint to the subgroup x_0/H generated. So it has rank 1. So we can embed it into $\mathbf{Q}/\mathbf{Z}$, $h': G/H \to \mathbf{Q}/\mathbf{Z}$, $h'(x_0) = 1/m$, and let $h: G \to \mathbf{Q}/\mathbf{Z}$ be such that $h(x) = h'(x/H)$. Since x_0/H has order m, $G = \langle H, x_0 \rangle$, and we see Range $h \subseteq \mathbf{Q}_m/\mathbf{Z}$ and clearly $H \subseteq \operatorname{Ker} h$.

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (a). Let $m = \Pi_{i<n} p_i^{k(i)}$, $k(i) \geqq 1$. It is easy to check that $Q_{p_i} \subseteq Q_m$, so Q_{p_i}/Z is a subgroup of Q_m/Z, and that Q_m/Z is the direct sum of Q_{p_i}/Z $(i < n)$, so let f_i be the projection from Q_m/Z onto Q_{p_i}/Z.

Let $h: G \to \mathbf{Q}_m/\mathbf{Z}$, $h_i = f_i h: G \to \mathbf{Q}_{p_i}/\mathbf{Z}$, I^* a pure subgroup of G isomorphic to Z. So by (c) there are homomorphisms $g_i: \operatorname{Ker} h_i \to I^* \cap \operatorname{Ker} h_i$, $g_i \restriction (I^* \cap \operatorname{Ker} h_i) =$ the identity. We want to define an appropriate g. For $x \in \operatorname{Ker} h$, obviously $x \in \operatorname{Ker} h_i$ (for $i < n$). Let $x_0 \in I^*$ generate it; so for each i, for some minimal $l(i) \geqq 0$, $p_i^{l(i)} x_0 \in \operatorname{Ker} h_i$. By elementary number theory there are natural numbers m_i such that

$$\sum_{l<n} m_i \prod_{\substack{j<n \\ j \neq i}} p_i^{l(i)} = 1.$$

Let us define $g: \operatorname{Ker} h \to I^*$ by

$$g(y) = \sum_{i<n} m_i \prod_{\substack{j<n \\ j \neq i}} p_i^{l(i)} g_i(y).$$

Note $\operatorname{Ker} h \subseteq \operatorname{Ker} h_i$ so $g(y)$ is well defined.

Note

$$\prod_{\substack{j<n \\ j \neq i}} p_i^{l(i)} y \in \operatorname{Ker} h_i,$$

so $g(y)$ is well defined.

Also, $g \restriction (I^* \cap \operatorname{Ker} h)$ is the identity as $g_i \restriction (I^* \cap \operatorname{Ker} h) \subseteq g_i \restriction (I^* \cap \operatorname{Ker} h_i)$ is the identity.

The last point we have to prove is that the range of g is $\subseteq I^* \cap \operatorname{Ker} h$. Obviously it is included in I^*, so we have to prove only $h(g(y)) = 0$. For this it suffices to prove

$$h\Big(\prod_{\substack{j<n \\ j\neq i}} p_i^{l(i)} g_i(y)\Big) = 0.$$

Now $g_i(y) = lp_i^{l(i)}x_0$ for some l, so it suffices to prove $\Pi_{j<n} p_i^{l(i)} h(x_0) = 0$ or $h(\Pi_{j<n} p_i^{l(i)} x_0) = 0$. But $\Pi_{j<n} p_i^{l(i)} x_0$ is clearly in $\operatorname{Ker} h_i$ for each i, hence is in $\operatorname{Ker} h$, as required.

Proof of Theorem 3.3. (ii) $\Rightarrow$ (i). We shall prove (b) of Claim 3.2. So let I^*, h, m be as there. Let $P = \{(f, I)$: I a pure subgroup of G of finite rank, $I^* \subseteq I$, $\operatorname{Dom} f = I \cap H$, $f: I \cap H \to I^* \cap H$ a homomorphism, $f \mid (I^* \cap H) =$ the identity$\}$, where $H = \operatorname{Ker} h$. P is ordered by: $(f, I) \leqq (f', I')$ if $f \subseteq f'$, $I \subseteq I'$.

As G is $\aleph_1$-free, also H is, so it is easy to check for $x \in H$ that $D_x = \{(f, I) \in P: x \in \operatorname{Dom} f\}$ is dense in P. So, as $|H| \leqq |G| < 2^{\aleph_0}$, by MA it suffices to prove that P satisfies the $\aleph_1$ chain condition. So let $(f_i, I_i) \in P$ $(i < \omega_1)$ be $\aleph_1$ conditions.

As we can replace them by any uncountable subfamily and increase, we can assume: I_i is freely generated by $a_1^i, \cdots, a_n^i$, $f_i(a_l^i) = s_l$ and $h(a_l^i) = t_l \in \mathbf{Q}_m/\mathbf{Z}$ $(l = 1, \cdots, n)$. Now by (ii) for m we can find $i < j < \omega_1$ satisfying (α), (β), (γ). So $h(a_l^i) = h(a_l^j) = t_l$ hence $a_l^i - a_l^j \in H$. By (β), there is a homomorphism f: $\langle I_i \cap H, I_j \cap H\rangle \to I^*$, $f \mid I^* \cap H = \mathrm{id}$, f_i, $f_j \subseteq f$. Clearly $\langle I_i \cap H, I_j \cap H\rangle \subseteq H$, and as $h(a_l^i - a_l^j) = 0$, $f(a_l^i - a_l^j) = 0$ we can extend f to

$$f': I' = \langle I_i \cap H, I_j \cap H, PC_H(a_1^i - a_1^j, \cdots, a_n^i - a_n^j)\rangle \to I^*,$$

$f \mid I^* =$ the identity. Let $I = PC_G(I_i, I_j)$. It suffices to prove $I \cap H = I'$; trivially $I' \subseteq I \cap H$. Now if $x \in I \cap H$, then by (α) $x = x_1 + x_2 + x_3$, $x_1 \in I_i$, $x_2 \in I_j$, $x_3 \in PC_G(a_1^i - a_1^j, \cdots, a_n^i - a_n^j)$. Let $x_2 = \Sigma k_l b_l^j$, and let $x_2' = \Sigma k_l b_l^i$, so $x_2' \in I_i$, and $x_2 - x_2' \in PC_G(a_1^i - a_1^j, \cdots, a_n^i - a_n^j)$. Let

$$x = (x_1 + x_2')) + (x_3 + (x_2 - x_2')), \qquad \text{so } x_1 + x_2' \in I_i,$$

$x_3 + (x_2 - x_2') \in PC_G(a_1^i - a_1^j, \cdots, a_n^i - a_n^j)$. So hence w.l.o.g. $x_2 = 0$. However as $H = \operatorname{Ker} h, i, j$ satisfy (γ) of (ii), clearly $PC_G(a_1^i - a_1^j, \cdots, a_n^i - a_n^j) \subseteq H$, so $x_3 \in H$, but as $x \in H$, also $x_1 \in H$. So

$$x = x_1 + x_3, \qquad x_1 \in I_i \cap H,$$
$$x_3 \in PC_G(a_1^i - a_1^j, \cdots, a_n^i - a_n^j) \cap H = PC_H(a_1^i - a_1^j, \cdots, a_n^i - a_n^j).$$

So $I' = I \cap H$, so we finish (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (ii). This proved as in the proof of "if G satisfies possibility III or II then it is Whitehead".

Not (iii) $\Rightarrow$ not (ii). This is proved as in the proof of Theorem 1.1.

CLAIM 3.4. *There is an* $\aleph_1$*-free group* G, $|G| = \aleph_1$, *satisfying* (ii) *of* 3.2 *which is not a Whitehead group. (So assuming* $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$, *there is a hereditarily separable non-Whitehead group of cardinality* $\aleph_1$.)

PROOF. Let $U \subseteq {}^{\omega}2$, $|U| = \aleph_1$; let G^0 be the free group generated by $\{x_\eta : \eta \in U\}$, G^* its divisible hull, $\{p_n : n < \omega\}$ indistinct primes, and $G \subseteq G^*$ be generated by

$$\{x_\eta : \eta \in U\} \cup \{(x_\eta - x_\nu)/p_n : \eta | n = \nu | n\}.$$

Its being non-Whitehead follows by 1.5. Now use 3.2 (you can use only (ii), (i), which was proved in detail).

THEOREM 3.5. ($2^{\aleph_0} < 2^{\aleph_1}$) *If* G *is hereditarily separable, then* G *is strongly* $\aleph_1$*-free. Moreover, if* $\bigcup_{i<\omega_1} G_i \subseteq G$, G_i *increasing continuous and countable, then* $\{\delta < \omega_1 : (\bigcup_{i<\omega_1} G_i)/G_\delta$ *is* $\aleph_1$*-free*$\}$ *is stationary.*

PROOF. Let $S = \{\delta < \omega_1 : \bigcup_{i<\omega_1} G_i/G_\delta$ is not free$\}$. We suppose S includes a closed unbounded set, and prove G is not hereditarily separable. This clearly suffices. We can assume w.l.o.g. $G = \bigcup_{i<\omega_1} G_i$, G_i a pure subgroup of G, and for $i \in S$, G_{i+1}/G_i is not free, has finite rank and has no subgroup of smaller rank which is not free, and G_0 has rank $\aleph_0$.

Denote $H = G_0$, choose $x_n \in H$, $H' \subseteq H$ such that $H = \langle H, x_0, \cdots, x_n, \cdots \rangle$, $x_0 \in H - H'$, $px_0 \in H'$, $x_n - px_{n+1} \in H'$. Now we define by induction on $i < \omega_1$, for every $\eta \in {}^i2$, a subgroup H_η of G_i such that:

(1) $\nu \triangleleft \eta$ implies $H_\nu \subseteq H_\eta$,

(2) $H_\eta \cap H = H'$, $G_i/H_\eta \cong \mathbf{Z}_p^{(\infty)}$,

(3) if $\delta \in S$, $\eta \in {}^\delta 2$, and $h_{\eta\hat{}\langle l\rangle}$ a projection from $H_{\eta\hat{}l}$ ($\subseteq G_{\delta+1}$) onto $\langle px_0 \rangle$ for $l = 0, 1$, then $h_{\eta\hat{}\langle 0\rangle} | H \neq h_{\eta\hat{}\langle 1\rangle} | H_\eta$.

This suffices: for every $\eta \in {}^{(\omega_1)}2$ let $H_\eta = \bigcup_{i<\omega_1} | H_{\eta|i} \subseteq G$; so if G is hereditarily separable for every such η there is a projection h_η from H_η onto $\langle px_0 \rangle$. As S includes a closed unbounded set (and $2^{\aleph_0} < 2^{\aleph_1}$) by Θ of [3], §6, for some $\delta \in S$, $\eta \in {}^\delta 2$, and $\nu_0, \nu_1 \in {}^{(\omega_1)}2$, $\eta\hat{}\langle l \rangle \triangleleft \nu_l$. So $h_1 | H_{\eta\hat{}\langle l\rangle}$ contradicts condition (3) above.

In the definition of H_η ($\eta \in {}^{(\omega_1)}2$) the cases $i = 0$, i limit and $i = j + 1$, $j \notin S$ cause no problem. For $i = j + 1$, $j \in S$ we have to take care of condition (3). This is similar to the proof of case (1) in the proof of Theorem 3.1. Let $\eta \in {}^j2$, and we define $H_{\eta\hat{}\langle l\rangle}$.

Let $\{z_1/G_j, \cdots, z_k/G_j\}$ be a maximal independent subset of G_i/G_j, and let I be a maximal subgroup of G_i such that $H_\eta \cup \{z_1, \cdots, z_{k-1}, pz_k\} \subseteq I$, $z_k + lx_0 \not\in I$ $(l = 0, 1, \cdots, p = 1)$. If I/H_η is not free we let $H_{\eta\hat{}\langle l\rangle} = \langle I, z_k + lx_0\rangle$ $(l = 0, 1)$: and as in subcase 1A of the proof of 3.1, (3) is satisfied. If I/H is free then as in subcase 1B of the proof of 3.1 we can find $t_k^\nu \in G_i$ $(\nu \in {}^k2,\ k < \omega)$ such that $t_0^\nu = z_k$, $pt_{k+1}^\nu - t_k^\nu \in I$, $t_{k+1}^{\nu\hat{}\langle 1\rangle} - t_{k+1}^{\nu\hat{}\langle 0\rangle} = x_0$. For each $\nu \in {}^\omega 2$ let $H_{\eta,\nu} = \langle H_\eta, t_0^{\nu|0}, t_1^{\nu|1}, \cdots\rangle$. Then we choose $\nu_0, \nu_1 \in {}^\omega 2$; let $H_{\eta\hat{}\langle l\rangle} = H_{\eta,\nu_l}$. So we have to prove that there are ν_0, ν_1 so that condition (3) holds. In fact for every ν_0 all but countably many $\nu_1 \in {}^\omega 2$ are suitable.

THEOREM 3.6. *Suppose G is torsion free, and for some finite set P^* of prime numbers and free $G^* \subsetneqq G$, G/G^* is a torsion group such that for no prime $p \not\in P^*$ is there an element of order p in G/G^*.*

Then G is hereditarily separable iff G is Whitehead.

PROOF. The "if" part appears in Nunke [9]. So suppose G is hereditarily separable, so we can assume $G = G' + \mathbf{Z}$. Clearly G is a Whitehead group if G' is a Whitehead group, and we shall prove the latter.

So let h be a homomorphism from H onto G' with kernel $\mathbf{Z} \subseteq H$. We can assume $G^* = (G' \cap G^*) + \mathbf{Z}$; let $\{a_i : i < \alpha\}$ freely generate $G' \cap G^*$.

We shall embed H into G, thus proving $\mathbf{Z}$ is a direct summand of H, hence h splits and we shall finish the proof.

Choose $b_i \in H$, $h(b_i) = a_i$, so clearly $\{b_i : i < \alpha\}$ generate freely a subgroup of H. Let n^* be the product of the primes in P^*.

Look at the family of embedding g, $\operatorname{Dom} g$ a subgroup of H including $\mathbf{Z} \cup \{b_i : i < \alpha\}$, $g(b_i) = a_i$, $g(x) = n^*x$ $(x \in \mathbf{Z})$. Clearly this family is non-empty and closed under unions of increasing chains, hence it contains a maximal member g^*. It suffices to prove $\operatorname{Dom} g^* = H$.

Note that for $m \in \mathbf{Z}$, $m \in \operatorname{Range} g^*$ implies m is divisible by n^* (otherwise in H, $1_{\mathbf{Z}}$ is divisible by some $n > 1$).

Suppose $\operatorname{Dom} g^* \neq H$. Clearly $H/\operatorname{Dom} g^*$ is torsion (as $\mathbf{Z} \cup \{b_i : i < \alpha\} \subseteq \operatorname{Dom} g^*$). So for some prime p and $x \in H$, $x \not\in \operatorname{Dom} g^*$, $px \in \operatorname{Dom} g^*$, and clearly it suffices to prove $g^*(px) \in G$ is divisible by p (in G).

For some natural numbers n, m and $y \in \mathbf{Z}$, and $i(l) < \alpha$, k_l integers $(l < m)$, we have

$$(1) \qquad npx = ny + \sum_{l=0}^{m-1} k_l b_{i(l)}$$

(this is possible as $px \in \operatorname{Dom} g^*$, and $\mathbf{Z}$ is a direct summand of $\operatorname{Dom} g^*$ since G is hereditarily separable, g^* an embedding into G; so if $npx = y_1 + \sum_{l=1}^{m-1} k_l b_{i(l)}$, y_1 is divisible by n as npx is (in $\operatorname{Dom} g^*$)),

$$g^*(npx) = nn^*y + \sum_{l=0}^{m-1} k_l a_{i(l)}, \tag{2}$$

and clearly

$$h(npx) = h(ny) + \sum_{l=0}^{m-1} k_l h(b_{i(l)}) = 0 + \sum_{l=0}^{m-1} k_l a_{i(l)} \in G', \tag{3}$$

hence

$$h(npx) = nph(x) \in G'. \tag{4}$$

As all groups here are torsion free, it suffices to prove $g^*(npx)$ is divisible by np (in G).

By equations (3), (4) it follows that $\Sigma_{l=0}^{m} k_l a_{i(l)}$ is divisible by np in G. So by equation (2) it suffices to prove n^*y is divisible by p in G. For this it suffices that p divides n^* or equivalently (by n^*'s definition) that $p \in P^*$. But this follows by the choice of G^*.

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